Assignment 2: BASICS-7

This assignment deals with repeat orbits and how the J_2 Earth deformation influences the repeat pattern. Two models are assumed throughout the assignment. In one model, the J_2 term only affects $\dot{\Omega}$. In the alternative model, the J_2 term affects $\dot{\Omega}$, $\dot{\omega}$ and $\dot{M}$.

1. Methodology used

1.1. Simple model

In the simple model, the repeat orbit is computed only with the Earth's rotation and J_2 causing a precession in the right ascension of the ascending node ($\dot{\Omega}$). The sum of these terms needs needs to be an integral number of complete rotations in order to have a repeat orbit (see eq 1).

$$j|\Delta L| = k2\pi \tag{1}$$

With j the amount of orbital periods, k the number of days and ΔL the rotation of the longitudinal shift. ΔL_1 contains the Earth rotation (see eq 3). In this term the orbit time is used to compute how far the earth has rotated after a complete orbit. Furthermore also the precession of Ω (see eq 4) contributes to the position of the ascending node.

$$\Delta L = \Delta \mathrm{L}_1 + \Delta \mathrm{L}_2 \tag{2}$$

$$\Delta \mathrm{L}_1 = -2\pi \frac{T}{D^*} \tag{3}$$

$$\Delta \mathrm{L}_2 = -\frac{3\pi J_2 R_e^2 \cos(i)}{a^2\left(1-e^2\right)^2} \tag{4}$$

$$\text{With: } T = 2\pi\sqrt{\frac{a^3}{\mu}} \text{ and } D^* = 86164s(1 \text{ sidereal day}) \tag{5}$$

In order to determine at what altitude, or what the semi-major axis of the orbit must be to have a specific repeat orbit, one solves eq 1 for the a term. For this, one needs a specific repeat orbit pattern (ea given k and j), eccentricity and inclination. This is done in Mathematica[1]. In this process all imaginary and impossible orbital altitudes are discarded (2 imaginary and 1 under the Earth's surface). This leaves one correct orbit altitude.

1.2. Complex model

In the complex model, the repeat orbit is computed with the Earth's rotation and J_2 having precession effects on $\dot{\Omega}$, the argument of perigee ($\dot{\omega}$) and mean anomaly ($\dot{M}$). In order to find how fast the satellites ascending node is rotating, the mean angular motion is defined (see 6).

$$n = \frac{j}{k}(\dot{L} - \dot{\Omega}) - (\dot{M} + \dot{\omega}) \tag{6}$$

The terms in this equation reflect what contributes to the motion if the ascending node. $\dot{L}$ is the earth rotation rate since the satellite is orbiting in the ECI frame. $\dot{\Omega}$ reflects the precession of the right ascension of the ascending node, $\dot{M}$ the change in mean anomaly and $\dot{\omega}$ the variation in argument of perigee (all J_2 effects). The specific equations are as follows (see OCDM[2] p81)

$$\dot{\Omega} = \frac{-3}{2} J_2 \sqrt{\mu} {R_E}^2 a^{-3.5} \text{Cos}[i] \left(1 - e^2\right)^{-2} \frac{180}{\pi} D^* \tag{7}$$

$$\dot{\omega} = \frac{3}{4} J_2 \sqrt{\mu} {R_E}^2 a^{-3.5} \left(5\text{Cos}[i]^2 - 1\right) \left(1 - e^2\right)^{-2} \frac{180}{\pi} D^* \tag{8}$$

$$\dot{M} = \frac{3}{4} J_2 \sqrt{\mu} {R_E}^2 a^{-3.5} \left(3\text{Cos}[i]^2 - 1\right) \left(1 - e^2\right)^{-1.5} \frac{180}{\pi} D^* \tag{9}$$

Knowing the mean angular motion, one can rework this to the orbital altitude using eq 10.

$$a = \sqrt[3]{\frac{\mu}{\left(\frac{n\pi}{D^* 180}\right)^2}} \tag{10}$$

However, in order to compute n, one already needs to know a. This solved by using an initial guess for a, and solving for n. Then a new value for the semi major axis is computed. this process is repeated until the desired accuracy is achieved. The initial guess in this document is assumed to be the Earth radius (R_e) and the accuracy was 1m.

2. Table verification

Numerically verify the contents of Table 2-13 of OCDM[2]

A copy of the Table 2-13 from OCDM has been provided here for convenience (See table 1).

Altitude(km)	Inclination(deg)	Period(min)	$\frac{\text{Orbits}}{\text{Day}}$	Repeat patern
817.14	28	101.24	14.0	14orbits/day
701.34	28	98.80	14.33	43orbits/3days
645.06	28	97.63	14.5	29orbits/2days
562.55	28	95.91	14.75	59orbits/4days
546.31	28	95.57	14.8	74orbits/5days
482.25	28	94.25	15.0	15orbits/day

Table 1: Table 2-13 of OCDM[2]: Regressive repeating ground track orbits

In order to verify if the values in table 1, the complex model of the repeat orbits was used (see eqs 6 - 10). Both the orbit altitude and the orbit time where determined. This is done with the inclination from the table, a zero eccentricity and the given orbits per day.

$h_{table} = 817140.m$	$h_{calc} = 817159.m$	$t_{table} = 101.24min$	$t_{calc} = 101.235min$
$h_{table} = 701340.m$	$h_{calc} = 702383.m$	$t_{table} = 98.8min$	$t_{calc} = 98.8229min$
$h_{table} = 645060.m$	$h_{calc} = 644894.m$	$t_{table} = 97.63min$	$t_{calc} = 97.6218min$
$h_{table} = 562550.m$	$h_{calc} = 562280.m$	$t_{table} = 95.91min$	$t_{calc} = 95.9043min$
$h_{table} = 546310.m$	$h_{calc} = 546025.m$	$t_{table} = 95.57min$	$t_{calc} = 95.5676min$
$h_{table} = 482250.m$	$h_{calc} = 481871.m$	$t_{table} = 94.25min$	$t_{calc} = 94.2425min$

Table 2: Orbit altitude and time comparison of computed and OCDM values

The start value for the semi-major axis was the Earth's radius and the stop condition for the iteration was 1m. The results are summarized in table 2.

One can see that the values computed differ from the tabulated values of OCDM[2] by a maximum of 1km (second entry). This is most probably due to differences in the constants used in the computations. Slight differences in the earth radius and gravitational parameter can induce these errors. However, these computations are still valid for rough initial estimates of the mission orbit, or if accuracies of about a km are acceptable for the mission.

3. Repeat orbit inventory

Make an inventory (tabular and in a plot) of the options for orbital parameters (a,i) for a 3-day zero-eccentricity Earth-repeat orbit, in the height regime between 200 and 1200 km, for j = 39, 40, 41, , 48. Do this separately for the approach with the J_2 perturbation on Ω only (as in ae3-803), and for the approach with the J_2 perturbation on Ω, ω and M (as discussed in the previous sheets). Discuss the results and the differences.

In order to make a table and plots of the orbital parameters for a given repeat orbit, a big lookup table was constructed. This lookup table has 10 rows, one for each j. For each j, the semi-major axis is computed both with J_2 effects on $\dot{\omega}$, $\dot{M}$ and without. This is done over the entire range of inclination (0°-180°) with steps of 5°. Then all the computed values where plotted (separately for both models). The model where J_2 only affects $\dot{\Omega}$ is graphed in figure 1. The model where J_2 affects $\dot{\Omega}$, $\dot{\omega}$ and $\dot{M}$ is graphed in figure 2. The actual values (condensed to 15° inclination steps) can be found in table 3.

What we can see is that the error between the two methods ranges according to the inclination and altitude. We see that the errors are large for 0° and for around 90°. Peaking to about 20km difference in orbit altitude. At around 60° the errors are the smallest. This is caused the J_2 deformation of the Earth. The J_2 effects on $\dot{\omega}$ and $\dot{M}$ are mostly at the locations where also the biggest errors between the two models occur. This can be checked by evaluating the cosine term in eq 8 and 9 ($x\ cos(i)^2 - 1$). These are maximal for a zero inclination and minimal for respectively 87° and 83°. Thus there the errors are the largest, as the simple model does not take these into account. Furthermore at around 60° the cosine term becomes zero and hence the two models become the same (Molniya orbit). The errors of the simple model do not make it unusable. It is still sufficient for a preliminary orbital analysis. Furthermore because the algorithms are a

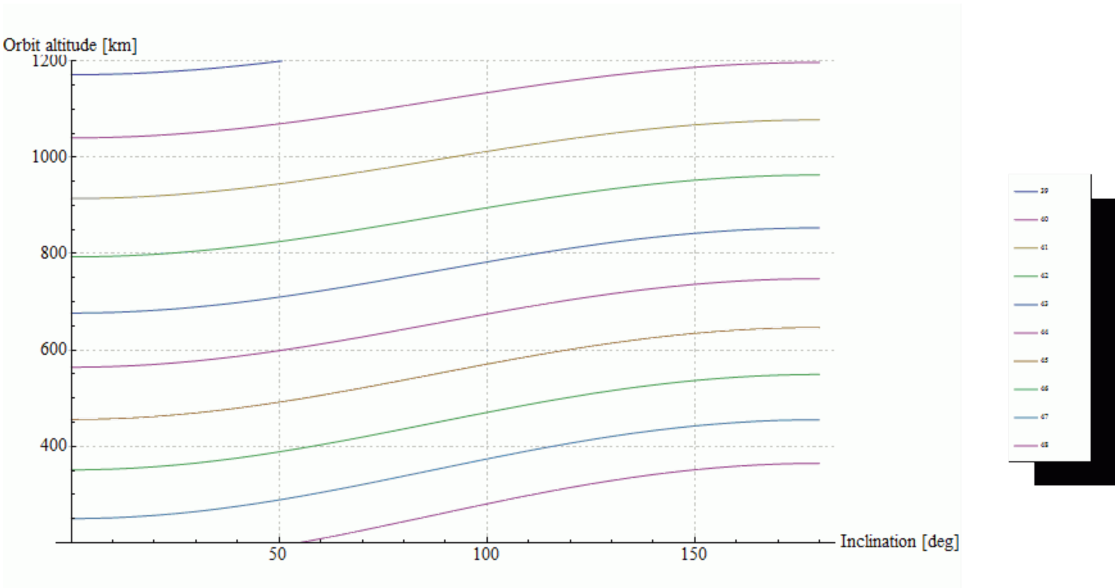

Figure 1: Orbital parameters for a given repeat interval [J_2 affects only $\dot{\Omega}$]

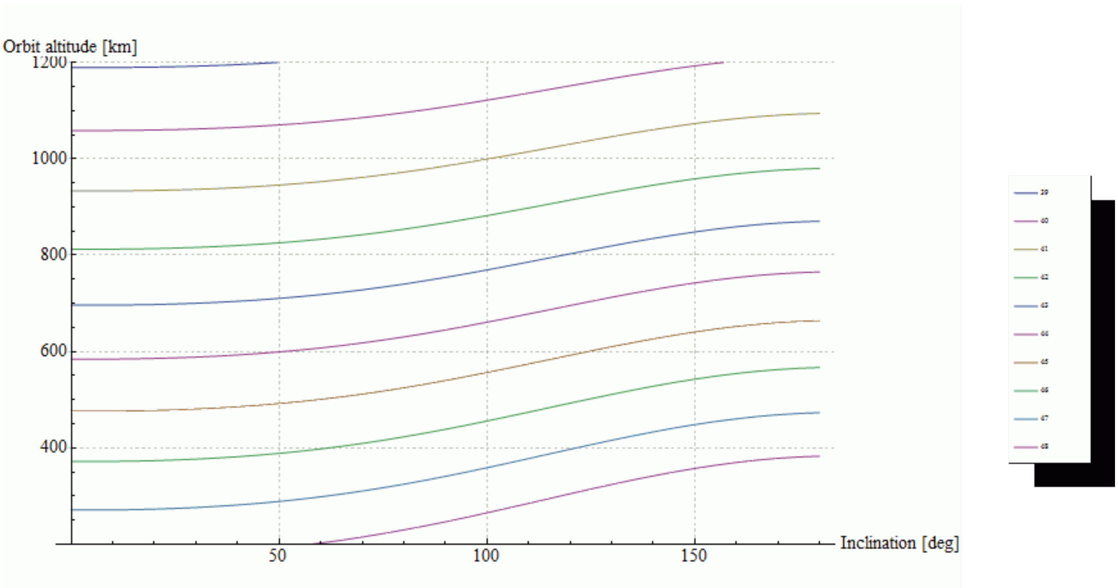

Figure 2: Orbital parameters for a given repeat interval [J_2 affects $\dot{\Omega}$, $\dot{\omega}$ and $\dot{M}$]

lot simpler it is easy to examine multiple possible orbits quickly. However if a more accurate model is required, the complex model can be used. This model however also has limits in its accuracy. Possible ways of improving are taking into account higher order J terms and possible 3^{rd} body perturbations.

	0	15	30	45	60	75	90	105	120	135	150	165	180
39	1189.41	1189.41	1190.45	1193.84	1200.18	1210.21	1224.34	1242.39	1263.31	1285.3	1306.04	1323.07	1334.26
39	1171.37	1174.04	1181.86	1194.23	1210.21	1228.63	1248.17	1267.49	1285.3	1300.45	1311.98	1319.19	1321.64
39													
40	1058.7	1058.7	1059.83	1063.49	1070.27	1080.9	1095.78	1114.65	1136.42	1159.23	1180.69	1198.27	1209.81
40	1040.3	1043.1	1051.28	1064.21	1080.9	1100.14	1120.53	1140.68	1159.23	1175.01	1187.01	1194.52	1197.07
40													
41	933.13	933.13	934.358	938.286	945.53	956.779	972.407	992.117	1014.76	1038.4	1060.57	1078.71	1090.6
41	914.375	917.301	925.848	939.351	956.779	976.841	998.099	1019.08	1038.4	1054.81	1067.3	1075.1	1077.75
41													
42	812.372	812.372	813.695	817.907	825.618	837.501	853.899	874.467	897.995	922.472	945.369	964.067	976.306
42	793.258	796.313	805.235	819.325	837.501	858.409	880.546	902.383	922.472	939.527	952.496	960.596	963.35
42													
43	696.118	696.118	697.54	702.044	710.234	722.767	739.951	761.392	785.815	811.139	834.768	854.029	866.621
43	676.643	679.831	689.137	703.828	722.767	744.536	767.568	790.27	811.139	828.846	842.304	850.707	853.563
43													
44	584.091	584.091	585.613	590.418	599.099	612.297	630.285	652.614	677.945	704.124	728.492	748.319	761.266
44	564.253	567.576	577.276	592.58	612.297	634.944	658.885	682.465	704.124	722.491	736.443	745.152	748.112
44													
45	476.035	476.035	477.661	482.773	491.959	505.837	524.644	547.877	574.128	601.173	626.285	646.681	659.984
45	455.832	459.293	469.396	485.326	505.837	529.378	554.243	578.713	601.173	620.207	634.657	643.674	646.738
45													
46	371.713	371.713	373.446	378.875	388.578	403.152	422.794	446.945	474.13	502.049	527.911	548.879	562.539
46	351.143	354.746	365.26	381.831	403.152	427.602	453.406	478.778	502.049	521.756	536.711	546.039	549.208
46													
47	270.91	270.91	272.751	278.505	288.739	304.024	324.518	349.603	377.733	406.535	433.152	454.695	468.715
47	249.969	253.718	264.652	281.878	304.024	329.401	356.158	382.445	406.535	426.923	442.385	452.027	455.302
47													
48	173.423	173.423	175.377	181.465	192.242	208.254	229.616	255.651	284.737	314.432	341.809	363.93	378.309
48	152.11	156.007	167.372	185.265	208.254	234.573	262.298	289.512	314.432	335.506	351.481	361.438	364.819

Computed orbit altitude [km] for j orbits (39-48) per 3 days, and a given inclination (0°-180°) and e=0.
The top value is for the model with J_2 effects on $\dot{\Omega}$, $\dot{\omega}$ and $\dot{M}$. For the bottom value J_2 only affects $\dot{\Omega}$

References

[1] W. Research. (2008) Mathematica edition: Version 7.0. Champaign, Illinois.

[2] J. R. Wertz, *Orbit & Constellation Design & Management*, second printing ed. El Segundo, California: Microcosm Press, 2009.

[3] R. Noomen, *AE4-879 Basics*, TUDelft Lecture Slides, 2010.

Additional information

Estimated work time:

$\sim$ 2h Studying theory + $\sim$ 3h making assignment + $\sim$ 4h writing report = $\sim$ 9h

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Version history

Version 1: Initial document

A. Mathematica source code

This is the source code in the form of one Mathematica[1] notebooks. The notebook (repeatOrbits.nb) is printed below:

repeatOrbits.nb | 1

Coefficients

In[4]:=
```
μ = 3.98600441 10^14 ;(* μ earth [m^3/s^2], see OCDM flap*)
J2 = 0.00108263 ;(* Earth J2 [-] see OCDM p65*)
RE = 6 378 138;(* Earth radius [m], see OCDM flap*)
D* = 86 164.10035; (* Sidereal day, [s] see OCDM p81*)
L = 360; (*[deg/sidereal day]*)
```

Calculate J_2on Ω only

In[8]:=
```
T[a_] := 2 π Sqrt[a^3/μ] (* Orbit time [s]*)
```

In[9]:=
```
ΔL1[a_] := -2 π T[a]/D* ; (* rad/rev *)
ΔL2[a_, e_, i_ ] = - (3 π J2 RE^2 Cos[i])/(a^2 (1 - e^2)^2); (* rad/rev *)
```

In[10]:=
```
(*Find the REAL solutions to the equation*)
FindASimple[e_, i_, j_, k_] := Select[
  Solve[k == (j Abs[ΔL1[a] + ΔL2[a, e, i]])/(2 π), a]
  , (Im[a /. #] == 0) &]

(*disgards any result in the earth (h<0)
 and returs the first solution (there is only one)*)
FindHSimple[e_, i_, j_, k_] := First[Select[
   a - RE /. FindASimple[e, i, j, k], # > 0 &
  ]]
```

Calculate J_2on Ω, ω and M

In[12]:=
```
Ω[a_, e_, i_, j_, k_] := -3/2 J2 Sqrt[μ] RE^2 a^-3.5 Cos[i] (1 - e^2)^-2 180/π D*
ω[a_, e_, i_, j_, k_] := 3/4 J2 Sqrt[μ] RE^2 a^-3.5 (5 Cos[i]^2 - 1) (1 - e^2)^-2 180/π D*
M[a_, e_, i_, j_, k_] := 3/4 J2 Sqrt[μ] RE^2 a^-3.5 (3 Cos[i]^2 - 1) (1 - e^2)^-1.5 180/π D*
n[a_, e_, i_, j_, k_] := j/k (L - Ω[a, e, i, j, k]) - (ω[a, e, i, j, k] + M[a, e, i, j, k])
```

In[16]:=
```
FindA[atemp_, i_, j_, k_] := (
  a = atemp;
  a2 = Infinity;
  While[Abs[a - a2] > 1,
   a2 = a;
   a = (μ / ((n[a,0,i,j,k] π)/(D*180))^2)^(1/3) // N;
   (*Print["h=",a-R_E, "m\t a=",a,"m"];*)
  ];
  a
 )
FindH[atemp_, i_, j_, k_] := a - R_E /. a → FindA[atemp, i, j, k];
```

Verify table data

In[18]:=
table =

Altitude (km)	Inclination (deg)	Period (min)	Orbits/Day	Repeat patern
817.14	28	101.24	14.0	14 orbits / day
701.34	28	98.80	14.33	43 orbits / 3 days
645.06	28	97.63	14.5	29 orbits / 2 days
562.55	28	95.91	14.75	59 orbits / 4 days
546.31	28	95.57	14.8	74 orbits / 5 days
482.25	28	94.25	15.0	15 orbits / day

;

We enter inclination and orbits per day, and see is the orbital period and altitude are the same as in the table

In[19]:=
```
For[row = 2, row < 8, row++,
  hTable = table[[row]][[1]] * 10^3; (*[m]*)
  iTable = table[[row]][[2]] Degree; (*[degree]*)
  tTable = table[[row]][[3]] ;
  j = table[[row]][[4]];
  a =.; (*Reset a*)
  hCalc = FindH[R_E, iTable, j, 1]; (*[-]*)
  tCalc = T[hCalc + R_E] / 60 // N;
  Print["h_table=", hTable, "m h_calc=",
   hCalc, "m t_table=", tTable, "min t_calc=", tCalc, "min"];
  (*Print["delta:",hTable-hCalc]*)
 ];
```


h_{table}=817 140.m h_{calc}=817 165.m t_{table}=101.24min t_{calc}=101.236min

h_{table}=701 340.m h_{calc}=702 389.m t_{table}=98.8min t_{calc}=98.823min

h_{table}=645 060.m h_{calc}=644 899.m t_{table}=97.63min t_{calc}=97.6219min

h_{table}=562 550.m h_{calc}=562 286.m t_{table}=95.91min t_{calc}=95.9044min

h_{table}=546 310.m h_{calc}=546 031.m t_{table}=95.57min t_{calc}=95.5677min

h_{table}=482 250.m h_{calc}=481 876.m t_{table}=94.25min t_{calc}=94.2426min

Make an invertory

In[20]:=
```
MkListSection[istart_, iend_, istep_, jstart_, jend_] := {
  inclinations =
   Mod[Range[istart,
     (iend - istart + istep) * (jend - jstart + 1) - istep, istep], (iend + istep)],

  repeats = Floor[Range[jstart, jend + 1 - 10^-20, istep / (iend - istart + istep)]]
 }
```

In[21]:=
```
(*Makes something like eg:*)
MkListSection[0, 10, 5, 2, 3] // TableForm
```

Out[21]//TableForm=
```
0  5  10  0  5  10
2  2  2   3  3  3
```

Compute all the values

In[22]:=
```
limitedCoef = {}; (* Holds results with J2 on Ω only *)
allCoef = {}; (* Holds results with J2 on Ω,ω and M *)
Module[{ranges, inclinations},
 For[j = 39, j ≤ 48, j++,
   ranges = MkListSection[0, 180, 15, j, j];
   inclinations = ranges[[1]];
   (* Compute simplified model *)
   a =.(*Reset a*)
    AppendTo[limitedCoef, {
      j,
      inclinations,
      Map[FindHSimple[0, # Degree , j, 3] / 1000 &, inclinations]
      (*This is the altitude in km !!*)
     }];
   (* Compute more complex model *)
   a =.(*Reset a*)
    AppendTo[allCoef, {
      j,
      inclinations,
      Map[FindH[RE, # Degree , j, 3] / 1000 &, inclinations]
      (*This is the altitude in km !!*)
     }];
  ];
];
```

Make the table for in the document

In[23]:=
```
list = {};
For[row = 1, row <= Dimensions[allCoef][[1]], row++,
  AppendTo[list, allCoef[[row, 3]]];
  AppendTo[list, limitedCoef[[row, 3]]];
  AppendTo[list, {}];
 ];
TableForm[list, TableHeadings → {Floor[Range[39, 48.5, 1 / 3]], allCoef[[1, 2]]}];
```

Make the plots of i vs a, with different orbit repeat lines

In[26]:=
```
Needs["PlotLegends`"]
PlotData[data2_, legend_] := Module[{data = {}},
  For[row = 1, row <= Dimensions[data2][[1]], row++,
    AppendTo[data, Transpose[data2[[row]]]];
   ]
   ListLinePlot[data, PlotRange → {200, 1200}, ImageSize → {1600, 800},
    GridLines → Automatic, GridLinesStyle → Directive[Opacity[0.5], Dashed],
    AxesLabel → {"Inclination [deg]", "Orbit altitude [km]"}, LabelStyle → 16,
    PlotLegend → legend, LegendSize → 0.7, LegendPosition → {1.1, -0.4}
   ]
 ]
```


In[28]:=
```
PlotData[allCoef[[All, {2, 3}]], allCoef[[All, 1]]]
PlotData[limitedCoef[[All, {2, 3}]], limitedCoef[[All, 1]]]
```

Null 


```
Needs["PlotLegends`"]
Module[{imin = 0, imax = 180, jmin = 39, jmax = 48, a = RE, k = 3},
 ShowLegend[ContourPlot[FindA[a, i Degree , j, k], {i, imin, imax}, {j, jmin, jmax},
   Contours → 50, ColorFunction → "Rainbow", ClippingStyle → Automatic],
  {ColorData["Rainbow"][1 - #1] &, 10,
   ToString[(FindA[a, imax Degree, jmin, k] - RE) / 1000],
   ToString[(FindA[a, imin Degree, jmax, k] - RE) / 1000],
   LegendPosition → {1, -0.7}, LegendSize → 1.5}]]
(*x: inclination, y: j orbits per 3 days, color: semi major axis*)
```

Out[31]=